2. Hausübung, **Statistische Physik**

abzugeben am Donnerstag, 27.10.2011

Aufgabe H3 *Paramagnetism* (6 Punkte)

Consider the spin system considered in class. We will use the Gaussian approximation for the multiplicity function:

$$g(N,s) \simeq g(N,0)\, e^{-2\frac{s^2}{N}}$$

In the presence of an external magnetic field B, the system's total energy is

$$U = -2smB$$

where m is a constant.
The fractional magnetization μ is the quantity

$$\mu = 2s/N.$$

a. Show that the temperature (in natural units) for this system is

$$\tau = -\frac{m^2B^2N}{U}.$$

b. What is the most likely value of μ if the system is in thermal equilibrium at temperature τ? Explain why this answer is certainly invalid when $\tau \leq mB$.

c. In the last class assignment, you computed that, at energy U, the probability of a given spin being up is

$$p(\uparrow) \simeq \frac{1}{1+e^{\frac{2U}{NmB}}}.$$

Express this probability as a function of the temperature τ of the whole system instead of the energy U, and show that

$$p(\downarrow)/p(\uparrow) = e^{-\delta U/\tau}$$

where δU is the energy increase resulting from flipping the spin from up to down. This ratio is called the *Boltzmann factor* for that spin which is in thermal contact with the large *reservoir* composed of all the other spins.

d. We are still interested in our single spin. Compute its average energy contribution $\overline{u}$ as a function of τ, given that spin up contributes $-mB$ and spin down contributes mB to the energy. Show that it is

$$\overline{u} = -mB\tanh(mB/\tau).$$

Aufgabe H4 *Quantum uniform states* (6 Punkte)

Recall that the von Neumann entropy of a quantum state ρ is

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho).$$

Consider a Hamiltonian operator H. Suppose that it has a discrete spectrum and P_i projects on the ith eigenspace, with energy (eigenvalue) $U_i \in \mathbb{R}$, i.e.

$$H = \sum_i U_i P_i.$$

This is the *spectral decomposition* of H, and P_i are *spectral projectors*. This expression is uniquely determined by the fact that the numbers U_i are distinct, and $P_1, P_2, \dots$ form a complete set of orthogonal projectors (PVM), i.e. they satisfy $P_i P_j = 0$ when $i \neq j$, and $\sum_i P_i = \mathbf{1}$. The value $\mathrm{Tr} P_i$ is the dimension of the ith eigenspace, also called the *degeneracy* of the eigenvalue U_i.

a. What is the state ρ which maximizes the entropy under the constraint that a measurement of the Hamiltonian will yield U_i with certainty? What is the value of that entropy? Sketch a proof of your answer.

   Hint: first show that we must have $\mathrm{Tr}(\rho P_i^{\perp}) = 0$, where $P_i^{\perp}$ projects on the subspace orthogonal to that of P_i, then express the trace as a sum of positive numbers.

b. Consider N independent copies of that system, with total Hamiltonian H_N being the sum of the individual Hamiltonians. Write down the spectral decomposition of H_N in terms of that of a single system H.

   Hint: express H_N first as a linear combination of the total PVM $\{P_{i_1} \otimes P_{i_2} \otimes \dots\}_{i_1, i_2, \dots = 1}^{\infty}$.